Regularization parameters for the self-force of a scalar particle in a general orbit about a Schwarzschild black hole

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Abstract

The interaction of a charged particle with its own field results in the *self-force* on the particle, which includes but is more general than the radiation reaction force. In the vicinity of the particle in curved spacetime, one may follow Dirac and split the retarded field of the particle into two parts, (1) the singular source field, $\sim q/r$, and (2) the regular remainder field. The singular source field exerts no force on the particle, and the self-force is entirely caused by the regular remainder. We describe an elementary multipole decomposition of the singular source field which is an important step in the calculation of the self-force on a scalar-charged particle orbiting a Schwarzschild black hole.

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I. INTRODUCTION

According to the equivalence principle in general relativity, a particle of infinitesimal mass orbits a black hole of large mass along a geodesic worldline Γ in the background spacetime determined by the large mass alone. For a particle of small but finite mass, the orbit is no longer a geodesic in the background of the large mass because the particle perturbs the spacetime geometry. This perturbation due to the presence of the smaller mass modifies the orbit of the particle from an original geodesic in the background. The difference of the actual orbit from a geodesic in the background is said to result from the interaction of the moving particle with its own gravitational field, which is called a *self-force* [1].

Historically, Dirac [2] first gave the analysis of the self-force for the electromagnetic field of a particle in flat spacetime. He was able to approach the problem in a perturbative scheme by allowing the particle's size to remain finite and invoking the conservation of the stress-energy tensor inside a narrow world tube surrounding the particle's worldline. Dewitt and Brehme [3] extended Dirac's problem to curved spacetime. Mino, Sasaki, and Tanaka [4] generalized it for the gravitational field self-force. Quinn and Wald [5] and Quinn [6] worked out similar schemes for the gravitational, electromagnetic, and scalar field self-forces by taking an axiomatic approach.

In Dirac's [2] flat spacetime problem, the retarded field is decomposed into two parts: (i) The first part is the "mean of the advanced and retarded fields" which is a solution of the inhomogeneous field equation resembling the Coulomb q/r piece of the scalar potential near the particle. (ii) The second part is a "radiation" field which is a homogeneous solution of Maxwell's equations. Dirac describes the self-force as the interaction of the particle with the radiation field, a well-defined solution of the vacuum field equations.

In the analyses of the self-force in curved spacetime, the Hadamard form of Green's function [3] is employed to describe the retarded field of the particle. Traditionally, taking the scalar field case for example, the retarded Green's function $G^{\rm ret}(p,p')$ is divided into *direct* and *tail* parts: (i) The direct part has support only on the past null cone of the field point p. (ii) The tail part has support inside the past null cone due to the presence of the curvature of spacetime. Accordingly, the self-force on the particle consists of two pieces: (i) The first piece comes from the direct part of the field and the acceleration of the worldline in the background geometry; this corresponds to Abraham-Lorentz-Dirac (ALD) force in flat spacetime. (ii) The second piece comes from the tail part of the field and is present in curved spacetime. Thus, the description of the self-force in curved spacetime reduces to Dirac's result in the flat spacetime limit. In this approach, the self-force is considered to result via

$$\mathcal{F}_a = q\nabla_a\psi, \tag{1}$$

from the interaction of the particle with the quantity [1]

$$\begin{aligned}\psi^{\rm self} &= \psi^{\rm ret} - \psi^{\rm dir} \\ &= -\left[\frac{qu(p,p')}{2\dot\sigma}\right]_{\tau_{\rm ret}}^{\tau_{\rm adv}} - q\int_{-\infty}^{\tau_{\rm ret}} v\,[p,p'(\tau')]\,d\tau'.\end{aligned} \tag{2}$$

The quantities u, v and σ are familiar from the Hadamard expansion of a Green's function [3].

Although this traditional approach provides adequate methods to compute the self-force, it does not share the physical simplicity of Dirac's analysis where the force is described

entirely in terms of an identifiable, vacuum solution of the field equations: unlike Dirac's radiation field, the $\psi^{\rm self}$ in Eq. (2) is not a solution of the vacuum field equation $\nabla^2\psi = 0$. Moreover, the integral term in $\psi^{\rm self}$ comes from the tail part of the Green's function and is generally not differentiable on the worldline if the Ricci scalar of the background is not zero (similarly, the electromagnetic potential $A_a^{\rm tail}$ and the gravitational metric perturbation $h_{ab}^{\rm tail}$ are not differentiable at the point of the particle unless $\left(R_{ab} - \frac{1}{6}g_{ab}R\right)u^b$ and $R_{cadb}u^cu^d$, respectively are zero in the background [7]). Thus, some version of averaging process must be invoked to make sense of the self-force.

In this paper we use an alternative method to split the retarded field $\psi^{\rm ret}$ in curved spacetime which is similar to Dirac's and suggested by Ref. [1]: (i) The *singular source field* $\psi^{\rm S}$ is an inhomogeneous field similar to the tidally distorted Coulomb field and exerts no force on the particle. (ii) The *regular remainder field* $\psi^{\rm R}$ is a homogeneous solution of the field equation, analogous to Dirac's radiation field, and is entirely responsible for the self-force. This alternative split is reviewed briefly in Section II.

In Section III we give a brief overview of the mode-sum decomposition scheme to evaluate the self-force [8]. We consider a particle with a scalar charge q in general motion about a Schwarzschild black hole. A spherical harmonic decomposition provides the multipole components of of both $\psi^{\rm ret}$ and $\psi^{\rm S}$. Then, the mode by mode sum of the difference of these components determines $\psi^{\rm R}$ and, thence, the self-force. The multipole components of $\psi^{\rm ret}$ can be determined numerically while the multipole components of $\psi^{\rm S}$ are derived analytically. In particular, the multipole moments of $\psi^{\rm S}$ are generically referred to as *regularization parameters* [8]. This paper focuses on the analytical task of finding these regularization parameters. Our analytical results are summarized at the end of the Section in Eqs. (12)-(19). These results are in agreement with those of Barack, Ori, Mino, Nakano, and Sasaki [9, 10, 11].

The description of $\psi^{\rm S}$ becomes particularly simple in a specially chosen co-moving frame: the THZ normal coordinates, introduced by Thorne and Hartle [12] and extended by Zhang [13], are locally inertial on a geodesic. In Section IV we obtain a simple form for $\psi^{\rm S}$ using the THZ coordinates and then re-express it in terms of the background Schwarzschild coordinates.

Section V outlines our derivation of the regularization parameters which, while not elementary, appears to us to be more compact than the derivations of others [9, 10].

Appendix A provides some mathematical details concerning the hypergeometric functions and the different representations of the regularization parameters in connection with them.

Notation: $(t,\, r,\, \theta,\, \phi)$ are the usual Schwarzschild coordinates. The particle moves along a worldline Γ, parameterized by the proper time τ, The points p and p' refer to a field point and a source point, respectively, on the worldline Γ of the particle. In the coincidence limit $p \to p'$. The coordinates $(T,\, X,\, Y,\, Z)$ are intermediate coordinates derived from the Schwarzschild coordinates, while $(\mathcal{T},\, \mathcal{X},\, \mathcal{Y},\, \mathcal{Z})$ are the THZ coordinates attached to the worldline Γ of the particle with $\rho \equiv \sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}$.

II. DECOMPOSITION OF THE RETARDED FIELD

The recent analysis of the Green's function decomposition by Detweiler and Whiting [1] shows a method to split the retarded field into two parts

$$\psi^{\rm ret} = \psi^{\rm S} + \psi^{\rm R}, \tag{3}$$

where ψ^{S} and ψ^{R} are the singular source field and the regular remainder field, respectively. The source function for a point particle on the worldline Γ is

$$\varrho(p) = q \int (-g)^{-1/2} \delta^4(p - p'(\tau')) d\tau'. \tag{4}$$

The singular source field ψ^{S} is an inhomogeneous solution of the scalar field equation

$$\nabla^2 \psi = -4\pi \varrho \tag{5}$$

in the neighborhood of the particle, just as ψ^{ret} is. And ψ^{S} is determined in the neighborhood of the particle's worldline entirely by local analysis. ψ^{R}, defined by Eq. (3), is then necessarily a homogeneous solution and is therefore expected to be differentiable on Γ. According to Ref. [1], ψ^{R} will formally give the correct self-force when substituted on the right hand side of Eq. (1) in place of ψ^{self}. In this paper we adopt this decomposition, and determine an analytical approximation, via a multipole expansion, of ψ^{S}, which is to be subtracted from ψ^{ret} for an explicit computation of the self-force.

III. MODE-SUM DECOMPOSITION AND REGULARIZATION PARAMETERS

By Eq. (1) the self-force can be formally evaluated from

$$\begin{aligned} \mathcal{F}^{\mathrm{self}}_a &= \lim_{p \to p'} \left[\mathcal{F}^{\mathrm{ret}}_a(p) - \mathcal{F}^{\mathrm{S}}_a(p) \right] = \mathcal{F}^{\mathrm{R}}_a(p') \\ &= q \lim_{p \to p'} \nabla_a (\psi^{\mathrm{ret}} - \psi^{\mathrm{S}}) = q \nabla_a \psi^{\mathrm{R}}, \end{aligned} \tag{6}$$

where p' is the event on Γ where the self-force is to be determined and p is an event in the neighborhood of p'. For use of this equation, both $\mathcal{F}^{\mathrm{ret}}_a(p)$ and $\mathcal{F}^{\mathrm{S}}_a(p)$ would be expanded into multipole ℓ-modes, with $\mathcal{F}^{\mathrm{ret}}_{\ell a}(p)$ determined numerically.

For the Schwarzschild spacetime, the source function $\varrho(p)$ is expanded in terms of spherical harmonics, and a similar expansion for ψ^{ret} is

$$\psi^{\mathrm{ret}} = \sum_{\ell m} \psi^{\mathrm{ret}}_{\ell m}(r,t) Y_{\ell m}(\theta, \phi), \tag{7}$$

where $\psi^{\mathrm{ret}}_{\ell m}(r,t)$ is found numerically. The individual components $\psi^{\mathrm{ret}}_{\ell m}$ in this expansion are finite at the location of the particle even though their sum is singular. Then, the ℓ component $\mathcal{F}^{\mathrm{ret}}_{\ell a}$ is finite

$$\mathcal{F}^{\mathrm{ret}}_{\ell a} = q \nabla_a \sum_m \psi^{\mathrm{ret}}_{\ell m} Y_{\ell m}. \tag{8}$$

The singular source field ψ^{S} is determined analytically in the neighborhood of the particle's worldline via local analysis (see Section IV) and its mode-sum decomposition provides

$$\mathcal{F}^{\mathrm{S}}_{\ell a} = q \nabla_a \sum_m \psi^{\mathrm{S}}_{\ell m} Y_{\ell m}, \tag{9}$$

which is also finite at the location of the particle. Eqs. (6), (8) and (9) now imply that

$$\begin{aligned} \mathcal{F}^{\mathrm{self}}_a &= \sum_\ell \lim_{p \to p'} \left[\mathcal{F}^{\mathrm{ret}}_{\ell a}(p) - \mathcal{F}^{\mathrm{S}}_{\ell a}(p) \right] \\ &= q \sum_\ell \lim_{p \to p'} \nabla_a \sum_m (\psi^{\mathrm{ret}}_{\ell m} - \psi^{\mathrm{S}}_{\ell m}) Y_{\ell m} \end{aligned} \tag{10}$$

evaluated at the location of the particle.

We follow Barack and Ori [8] in defining the regularization parameters, except that the singular source field $\psi^{\rm S}$ is used in place of $\psi^{\rm dir}$,

$$\lim_{p\to p'} \mathcal{F}^{\rm S}_{\ell a} = \left(\ell+\frac{1}{2}\right) A_a + B_a + \frac{C_a}{\ell+\frac{1}{2}} + O(\ell^{-2}). \tag{11}$$

In Section V these regularization parameters are derived from the multipole components of $\nabla_a \psi^{\rm S}$ evaluated at the source point,

$$A_t = \mathrm{sgn}(\Delta) \frac{q^2}{r_{\rm o}^2} \frac{\dot{r}}{1+J^2/r_{\rm o}^2}, \tag{12}$$

$$A_r = -\mathrm{sgn}(\Delta) \frac{q^2}{r_{\rm o}^2} \frac{E\left(1-\frac{2M}{r_{\rm o}}\right)^{-1}}{1+J^2/r_{\rm o}^2}, \tag{13}$$

$$A_\phi = 0, \tag{14}$$

$$B_t = \frac{q^2}{r_{\rm o}^2} E\dot{r}\left[\frac{F_{3/2}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}} - \frac{3F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right], \tag{15}$$

$$B_r = \frac{q^2}{r_{\rm o}^2}\left\{-\frac{F_{1/2}}{\left(1+J^2/r_{\rm o}^2\right)^{1/2}} + \frac{\left[1-2\left(1-\frac{2M}{r_{\rm o}}\right)^{-1}\dot{r}^2\right]F_{3/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{3/2}} + \frac{3\left(1-\frac{2M}{r_{\rm o}}\right)^{-1}\dot{r}^2 F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right\}, \tag{16}$$

$$B_\phi = \frac{q^2}{J}\dot{r}\left[\frac{F_{1/2}-F_{3/2}}{\left(1+J^2/r_{\rm o}^2\right)^{1/2}} + \frac{3(F_{5/2}-F_{3/2})}{2\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\right], \tag{17}$$

$$C_t = C_r = C_\phi = 0, \tag{18}$$

$$A_\theta = B_\theta = C_\theta = 0, \tag{19}$$

where $\Delta \equiv r - r_{\rm o}$, $E \equiv -u_t = (1-2M/r_{\rm o})\,(dt/d\tau)_{\rm o}$ (τ: proper time) and $J \equiv u_\phi = r_{\rm o}^2\,(d\phi/d\tau)_{\rm o}$ are the conserved energy and angular momentum, respectively, and $\dot{r} \equiv u^r = (dr/d\tau)_{\rm o}$. The subscript $_{\rm o}$ denotes evaluation at the location of the particle. Also, shorthand for the hypergeometric function is $F_p \equiv {}_2F_1\left(p, \frac{1}{2}; 1; J^2/(r_{\rm o}^2+J^2)\right)$ (see Appendix A for more details about the hypergeometric functions and the representations of the regularization parameters in terms of them).

IV. DETERMINATION OF $\psi^{\rm S}$ IN LOCALLY INERTIAL COORDINATES

In the vicinity of an event p' on a timelike worldline Γ, physics is most easily described in terms of locally inertial coordinates, where the time coordinate $\mathcal{T}$ on Γ is equal to the the proper time, and the orthogonal Cartesian-like spatial coordinates are ($\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$) and centered on Γ. At p', with locally inertial coordinates the spacetime metric is just the flat Minkowski metric, and all of the Christoffel symbols vanish. Locally inertial coordinates are not unique and have an ambiguity at $O(\rho^3)$, where $\rho^2 = \sqrt{\mathcal{X}^2+\mathcal{Y}^2+\mathcal{Z}^2}$. For example, differences of $O(\rho^3)$ distinguish Riemann normal from Fermi normal coordinates [14]. For our purposes a locally inertial coordinate system introduced by Thorne and Hartle [12] and later extended by Zhang [13] is particularly advantageous. In these *THZ* coordinates

Detweiler, Messaritaki, and Whiting [7] (cited henceforth as Paper I) show that the scalar wave equation takes a simple form and also that

$$\psi^{\mathrm{S}} = q/\rho + O(\rho^2/\mathcal{R}^3), \tag{20}$$

where $\mathcal{R}$ represents a length scale of the background geometry (the smallest of the radius of curvature, the scale of inhomogeneities and time scale for changes in curvature along Γ). Approximation (20) is accurate enough for self-force regularization because

$$\nabla_a \psi^{\mathrm{S}} = \nabla_a (q/\rho) + O(\rho/\mathcal{R}^3), \tag{21}$$

and the $O(\rho/\mathcal{R}^3)$ remainder vanishes at the particle.

For the derivation of the regularization parameters from the multipole components of $\nabla_a \psi^{\mathrm{S}}$, requires that ρ in Eq. (20) be expressed in terms of the coordinates of the background geometry. Thus, we look for the relationship between the original Schwarzschild coordinates $(t,\, r,\, \theta,\, \phi)$ and the THZ coordinates $(\mathcal{T},\, \mathcal{X},\, \mathcal{Y},\, \mathcal{Z})$ associated with an event p' on Γ. However, in Section V we surprisingly find that *any* locally inertial coordinate system is sufficient to determine the regularization parameters quoted in Eqs. (12)-(19), which are actually independent of the $O(\rho^3)$ terms in the THZ coordinates. An elementary discussion in Weinberg [15] determines this coordinate transformation through terms of $O(\rho^2)$ in two steps:

(*i*) Find inertial coordinates X^A in the neighborhood of the event p' on Γ in terms of a Taylor expansion of the Schwarzschild coordinates x^a about p', where the Schwarzschild coordinates at p' are x^a_{o} and the subscript $_{\mathrm{o}}$ denotes evaluation at p'. Weinberg's [15] Eq. (3.2.12) is

$$X^A = X^A_{\mathrm{o}} + M^A{}_a (x^a - x^a_{\mathrm{o}}) + \frac{1}{2} M^A{}_a \, \Gamma^a_{bc}\big|_{\mathrm{o}} (x^b - x^b_{\mathrm{o}})(x^c - x^c_{\mathrm{o}}) + O[(x - x_{\mathrm{o}})^3], \tag{22}$$

where we may choose $X^A_{\mathrm{o}} = 0$ and $M^A{}_a = \mathrm{diag}\left[M^T{}_t,\, M^X{}_r,\, M^Y{}_\phi,\, M^Z{}_\theta\right]$ for convenience as this choice recenters and rescales the Schwarzschild coordinates to $T = M^T{}_t (t - t_{\mathrm{o}})$, $X = M^X{}_r (r - r_{\mathrm{o}})$, $Y = M^Y{}_\phi (\phi - \phi_{\mathrm{o}})$, $Z = M^Z{}_\theta (\theta - \theta_{\mathrm{o}})$.

(*ii*) Boost X^A with u^A, the particle's four-velocity at p' as measured in this Cartesian frame, to obtain the final coordinates $\mathcal{X}^{A'}$;

$$\begin{aligned}
\mathcal{X}^{A'} &= \Lambda^{A'}{}_A X^A \\
&= \Lambda^{A'}{}_A \left[M^A{}_a (x^a - x^a_{\mathrm{o}}) + \frac{1}{2} M^A{}_a \, \Gamma^a_{bc}\big|_{\mathrm{o}} (x^b - x^b_{\mathrm{o}})(x^c - x^c_{\mathrm{o}}) \right] + O[(x - x_{\mathrm{o}})^3],
\end{aligned} \tag{23}$$

where

$$\Lambda^{A'}{}_A = \begin{bmatrix} u^T & -u^X & -u^Y & -u^Z \\ & 1 + (u^T - 1)(u^X)^2/u^2 & (u^T - 1) u^X u^Y / u^2 & (u^T - 1) u^X u^Z / u^2 \\ & & 1 + (u^T - 1)(u^Y)^2/u^2 & (u^T - 1) u^Y u^Z / u^2 \\ & \mathrm{sym} & & 1 + (u^T - 1)(u^Z)^2/u^2 \end{bmatrix} \tag{24}$$

is the upper half of the symmetric matrix $\Lambda^{A'}{}_A$ with $u^2 \equiv (u^X)^2 + (u^Y)^2 + (u^Z)^2$ [16].

With the choice of $M^T{}_t = (1-2M/r_{\rm o})^{1/2}$, $M^X{}_r = (1-2M/r_{\rm o})^{-1/2}$, $M^Y{}_\phi = r_{\rm o}\sin\theta_{\rm o}$ and $M^Z{}_\theta = -r_{\rm o}$, it follows that

$$\begin{aligned} g^{A'B'} &= g^{ab}\frac{\partial \mathcal{X}^{A'}}{\partial x^a}\frac{\partial \mathcal{X}^{B'}}{\partial x^b} \\ &= \eta^{A'B'} + O[(x-x_{\rm o})^2],\ x^a \to x^a_{\rm o}, \end{aligned} \tag{25}$$

so that

$$\frac{\partial g^{A'B'}}{\partial \mathcal{X}^{C'}} = O[(x-x_{\rm o})],\ x^a \to x^a_{\rm o}, \tag{26}$$

Eqs. (25) and (26) are the desired locally inertial features for a particle in the Schwarzschild geometry at event $x^a_{\rm o}$ with four-velocity u^a.

To simplify the calculations, we confine the particle's orbit to the equatorial plane $\theta_{\rm o} = \pi/2$ and have

$$M^A{}_a = \mathrm{diag}\left[f^{1/2},\ f^{-1/2},\ r_{\rm o},\ -r_{\rm o}\right], \tag{27}$$

where $f \equiv (1-\frac{2M}{r_{\rm o}})$. This constraint to the equatorial plane makes $u^Z = 0$ and we rewrite u^A

$$u^A \equiv (u^T,\ u^X,\ u^Y,\ u^Z) = \left(f^{-1/2}E,\ f^{-1/2}\dot{r},\ \frac{J}{r_{\rm o}},\ 0\right), \tag{28}$$

in terms of the Schwarzschild coordinates and the constants of motion: $E \equiv -u_t = f\,(dt/d\tau)_{\rm o}$ and $J \equiv u_\phi = r^2_{\rm o}\,(d\phi/d\tau)_{\rm o}$ are the conserved energy and angular momentum, respectively, and $\dot{r} \equiv u^r = (dr/d\tau)_{\rm o}$. From this it follows that $u^2 = f^{-1}E^2 - 1$ and we have

$$\Lambda^{A'}{}_A = \begin{bmatrix} f^{-1/2}E & -f^{-1/2}\dot{r} & -J/r_{\rm o} & 0 \\ & 1+\dot{r}^2/(f^{1/2}E+f) & J\dot{r}/[r_{\rm o}(E+f^{1/2})] & 0 \\ & & 1+J^2/[r^2_{\rm o}(f^{-1/2}E+1)] & 0 \\ & \text{sym} & & 1 \end{bmatrix}. \tag{29}$$

Now we are able to express ρ^2 in terms of the Schwarzschild coordinates using Eq. (23) and obtain

$$\begin{aligned} \rho^2 = \mathcal{X}^I\mathcal{X}_I &= \delta_{IJ}\Lambda^I{}_C\Lambda^J{}_D M^C{}_c M^D{}_d\left[(x^c-x^c_{\rm o})(x^d-x^d_{\rm o}) + \Gamma^c_{ab}\big|_{\rm o}(x^a-x^a_{\rm o})(x^b-x^b_{\rm o})(x^d-x^d_{\rm o})\right] \\ &\quad + O[(x-x_{\rm o})^4], \end{aligned} \tag{30}$$

where $I,\ J = 1,\ 2,\ 3$. Then, after a substitution from Eqs. (27) and (29), ρ^2 becomes

$$\begin{aligned} \rho^2 &= (E^2-f)(t-t_{\rm o})^2 - \frac{2E\dot{r}}{f}(t-t_{\rm o})(r-r_{\rm o}) - 2EJ(t-t_{\rm o})(\phi-\phi_{\rm o}) \\ &\quad + \frac{1}{f}\left(1+\frac{\dot{r}^2}{f}\right)(r-r_{\rm o})^2 + \frac{2J\dot{r}}{f}(r-r_{\rm o})(\phi-\phi_{\rm o}) + (r^2_{\rm o}+J^2)(\phi-\phi_{\rm o})^2 + r^2_{\rm o}\left(\theta-\frac{\pi}{2}\right)^2 \\ &\quad - \frac{ME\dot{r}}{r^2_{\rm o}}(t-t_{\rm o})^3 + \frac{M}{r^2_{\rm o}}\left(-1+\frac{2E^2}{f}+\frac{\dot{r}^2}{f}\right)(t-t_{\rm o})^2(r-r_{\rm o}) + \frac{MJ\dot{r}}{r^2_{\rm o}}(t-t_{\rm o})^2(\phi-\phi_{\rm o}) \\ &\quad - \frac{ME\dot{r}}{f^2r^2_{\rm o}}(t-t_{\rm o})(r-r_{\rm o})^2 - \frac{2(r_{\rm o}-M)EJ}{fr^2_{\rm o}}(t-t_{\rm o})(r-r_{\rm o})(\phi-\phi_{\rm o}) \\ &\quad + r_{\rm o}E\dot{r}(t-t_{\rm o})(\phi-\phi_{\rm o})^2 + r_{\rm o}E\dot{r}(t-t_{\rm o})\left(\theta-\frac{\pi}{2}\right)^2 \end{aligned}$$

$$
-\frac{M}{f^2 r_o^2}\left(1+\frac{\dot{r}^2}{f}\right)(r-r_o)^3+\frac{(2r_o-5M)J\dot{r}}{f^2 r_o^2}(r-r_o)^2(\phi-\phi_o)
$$
$$
+r_o\left(1-\frac{\dot{r}^2}{f}+\frac{2J^2}{r_o^2}\right)(r-r_o)(\phi-\phi_o)^2+r_o\left(1-\frac{\dot{r}^2}{f}\right)(r-r_o)\left(\theta-\frac{\pi}{2}\right)^2
$$
$$
-r_o J\dot{r}(\phi-\phi_o)^3-r_o J\dot{r}(\phi-\phi_o)\left(\theta-\frac{\pi}{2}\right)^2+O[(x-x_o)^4]. \tag{31}
$$

The substitution of Eq. (31) into Eq. (20) approximates $\psi^{\rm S}$ in terms of the Schwarzschild coordinates and leads to the derivation of the regularization parameters in the next section.

In the above analysis the $O[(x-x_o)^3]$ term in Eq. (23) contributes to the $O[(x-x_o)^4]$ terms of ρ^2 in Eqs. (30) and (31). To the level of accuracy we desire for the mode-sum regularization parameters in this paper, that is to say, to the determination of C_a-terms, it is not necessary to specify the $O[(x-x_o)^4]$ term in ρ^2 and, hence, not necessary to specify the $O[(x-x_o)^3]$ terms in the spatial THZ coordinates $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$.

V. REGULARIZATION PARAMETERS FOR A GENERAL ORBIT OF THE SCHWARZSCHILD GEOMETRY

In Section IV, we have seen that an approximation to $\psi^{\rm S}$ is

$$
\psi^{\rm S}=q/\rho+O(\rho^2/\mathcal{R}^3). \tag{32}
$$

Following Paper I [7], the regularization parameters can be determined from evaluating the multipole components of $\partial_a(q/\rho)$ ($a=t$, r, θ, ϕ for the Schwarzschild background). The error, $O(\rho^2/\mathcal{R}^3)$ in the above approximation is disregarded since it gives no contribution to $\nabla_a\psi^{\rm S}$ as we take the "coincidence limit", $x^a\to x^a_o$, where x^a denotes a point in the vicinity of the particle and x^a_o the location of the particle in the Schwarzschild geometry.

In evaluating the multipole components of $\partial_a(q/\rho)$, singularities are expected with certain terms. To help identify those singularities, we introduce an order parameter ϵ which is to be set to unity at the end of the calculation: we attach ϵ^n to each $O[(x-x_o)^n]$ part of ρ^2 in Eq. (31) and re-express ρ^2 as

$$
\rho^2=\epsilon^2\mathcal{P}_{\rm II}+\epsilon^3\mathcal{P}_{\rm III}+\epsilon^4\mathcal{P}_{\rm IV}+O(\epsilon^5), \tag{33}
$$

where $\mathcal{P}_{\rm II}$, $\mathcal{P}_{\rm III}$, and $\mathcal{P}_{\rm IV}$ represent the quadratic, cubic and quartic order parts of ρ^2, respectively. Here we pretend that the quartic part $\mathcal{P}_{\rm IV}$ is also specified: this will help us to perform the structure analysis for C_a-terms later in Subsection V C when we prove that these regularization parameters always vanish.

We express $\partial_a\,(1/\rho)$ in a Laurent series expansion where every denominator of this expansion takes the form of $\mathcal{P}_{\rm II}^{n/2}$ ($n=3$, 5, 7). Thus, $\mathcal{P}_{\rm II}$ plays an important role in the multipole decomposition, and the quadratic part $\mathcal{P}_{\rm II}$, directly taken from Eq. (31), is not yet fully ready for this task. First, $\phi-\phi_o$ must be decoupled from $r-r_o$ so that each appears only as an independent complete square. Coupling between $t-t_o$ and $\phi-\phi_o$ does not create difficulty in the decomposition. Thus, we reshape the quadratic term of Eq. (31) into

$$
\begin{aligned}
\mathcal{P}_{\rm II} \;=\; & (E^2-f)(t-t_o)^2-\frac{2E\dot{r}r_o^2}{f\,(r_o^2+J^2)}(t-t_o)\Delta-2EJ(t-t_o)(\phi-\phi') \\
& +\frac{E^2 r_o^2}{f^2\,(r_o^2+J^2)}\Delta^2+\left(r_o^2+J^2\right)(\phi-\phi')^2+r_o^2\left(\theta-\frac{\pi}{2}\right)^2
\end{aligned} \tag{34}
$$

with

$$\phi' \equiv \phi_{\rm o} - \frac{J\dot{r}\Delta}{f\,(r_{\rm o}^2 + J^2)}, \tag{35}$$

where $\Delta \equiv r - r_{\rm o}$, and an identity $\dot{r}^2 = E^2 - f\,(1 + J^2/r_{\rm o}^2)$ is used for simplifying the coefficient of Δ^2. Here, taking the coincidence limit $\Delta \to 0$, we have $\phi' \to \phi_{\rm o}$. This same idea is found in Mino, Nakano, and Sasaki [9]. Also, for the multipole decomposition the quadratic part must be analytic and smooth over the entire two-sphere, and we write

$$\begin{aligned}
\mathcal{P}_{\rm II} \; = \; & (E^2 - f)(t - t_{\rm o})^2 - \frac{2E\dot{r}r_{\rm o}^2}{f\,(r_{\rm o}^2 + J^2)}(t - t_{\rm o})\Delta - 2EJ(t - t_{\rm o})\sin\theta\sin(\phi - \phi') \\
& + \frac{E^2 r_{\rm o}^2}{f^2\,(r_{\rm o}^2 + J^2)}\Delta^2 + (r_{\rm o}^2 + J^2)\sin^2\theta\sin^2(\phi - \phi') + r_{\rm o}^2\cos^2\theta \\
& + O[(x - x_{\rm o})^4].
\end{aligned} \tag{36}$$

Here we have used the elementary approximations $\phi - \phi' = \sin(\phi - \phi') + O[(\phi - \phi')^3]$ and $1 = \sin\theta + O[(\theta - \pi/2)^2]$.

To aid in the multipole decomposition we rotate the usual Schwarzschild coordinates by following the approach of Barack and Ori [10] such that the coordinate location of the particle is moved from the equatorial plane $\theta = \pi/2$ to the new polar axis. The new angles Θ and Φ defined in terms of the usual Schwarzschild angles are

$$\begin{aligned}
\sin\theta\cos(\phi - \phi') \; &= \; \cos\Theta \\
\sin\theta\sin(\phi - \phi') \; &= \; \sin\Theta\cos\Phi \\
\cos\theta \; &= \; \sin\Theta\sin\Phi.
\end{aligned} \tag{37}$$

Also, under this coordinate rotation, a spherical harmonic $Y_{\ell m}(\theta, \phi)$ becomes

$$Y_{\ell m}(\theta, \phi) = \sum_{m'=-\ell}^{\ell} \alpha^{\ell}_{mm'} Y_{\ell m'}(\Theta, \Phi), \tag{38}$$

where the coefficients $\alpha^{\ell}_{mm'}$ depend on the rotation $(\theta, \phi) \to (\Theta, \Phi)$ as well as on ℓ, m and m', and the index ℓ is preserved under the rotation [17]. As recognized in Ref. [10], there is a great advantage of using the rotated angles (Θ, Φ): after expanding $\partial_a(q/\rho)$ into a sum of spherical harmonic components, we take the coincidence limit $\Delta \to 0$, $\Theta \to 0$. Then, finally only the $m = 0$ components contribute to the self-force at $\Theta = 0$ because $Y_{\ell m}(0, \Phi) = 0$ for $m \neq 0$. Thus, the regularization parameters of Eq. (11) are just $(\ell, m = 0)$ spherical harmonic components of $\partial_a(q/\rho)$ evaluated at $x^a_{\rm o}$.

Now, with these rotated angles, $\mathcal{P}_{\rm II}$ is re-expressed as

$$\begin{aligned}
\mathcal{P}_{\rm II} \; = \; & (E^2 - f)(t - t_{\rm o})^2 - \frac{2E\dot{r}r_{\rm o}^2}{f\,(r_{\rm o}^2 + J^2)}(t - t_{\rm o})\Delta - 2EJ(t - t_{\rm o})\sin\Theta\cos\Phi \\
& + 2\left(r_{\rm o}^2 + J^2\right)\left(1 - \frac{J^2\sin^2\Phi}{r_{\rm o}^2 + J^2}\right)\left[\frac{r_{\rm o}^2 E^2 \Delta^2}{2f^2\,(r_{\rm o}^2 + J^2)^2\left(1 - \frac{J^2\sin^2\Phi}{r_{\rm o}^2 + J^2}\right)} + 1 - \cos\Theta\right] \\
& + O[(x - x_{\rm o})^4],
\end{aligned} \tag{39}$$

where the elementary approximation $\sin^2\Theta = 2(1-\cos\Theta) + O(\Theta^4)$ is used. We may now define

$$\begin{aligned}\tilde{\rho}^2 &\equiv (E^2 - f)(t-t_\mathrm{o})^2 - \frac{2E\dot{r}r_\mathrm{o}^2}{f\left(r_\mathrm{o}^2 + J^2\right)}(t-t_\mathrm{o})\Delta - 2EJ(t-t_\mathrm{o})\sin\Theta\cos\Phi \\ &+2\left(r_\mathrm{o}^2 + J^2\right)\left(1 - \frac{J^2\sin^2\Phi}{r_\mathrm{o}^2+J^2}\right)\left[\frac{r_\mathrm{o}^2E^2\Delta^2}{2f^2\left(r_\mathrm{o}^2+J^2\right)^2\left(1-\frac{J^2\sin^2\Phi}{r_\mathrm{o}^2+J^2}\right)} + 1 - \cos\Theta\right]. \quad (40)\end{aligned}$$

In particular, when fixing $t = t_\mathrm{o}$, we define

$$\tilde{\rho}_\mathrm{o}^2 \equiv \tilde{\rho}^2\Big|_{t=t_\mathrm{o}} = 2\left(r_\mathrm{o}^2 + J^2\right)\chi\left(\delta^2 + 1 - \cos\Theta\right) \quad (41)$$

with

$$\chi \equiv 1 - \frac{J^2\sin^2\Phi}{r_\mathrm{o}^2 + J^2} \quad (42)$$

and

$$\delta^2 \equiv \frac{r_\mathrm{o}^2E^2\Delta^2}{2f^2\left(r_\mathrm{o}^2+J^2\right)^2\chi}. \quad (43)$$

Now we rewrite Eq. (33) by replacing the original quadratic part $\mathcal{P}_\mathrm{II}$ with $\tilde{\rho}^2$,

$$\rho^2 = \epsilon^2\tilde{\rho}^2 + \epsilon^3\mathcal{P}_\mathrm{III} + \epsilon^4\mathcal{P}_\mathrm{IV} + O(\epsilon^5), \quad (44)$$

where $\mathcal{P}_\mathrm{IV}$ now includes the additional quartic order terms that have resulted from the replacement of $\mathcal{P}_\mathrm{II}$ by $\tilde{\rho}^2$. A Laurent series expansion of $\partial_a(1/\rho)|_{t=t_\mathrm{o}}$ is

$$\partial_a\left(\frac{1}{\rho}\right)\Bigg|_{t=t_\mathrm{o}} = -\frac{1}{2}\frac{\partial_a\left(\tilde{\rho}^2\right)|_{t=t_\mathrm{o}}}{\tilde{\rho}_\mathrm{o}^3}\epsilon^{-2} + \left\{-\frac{1}{2}\frac{\partial_a\mathcal{P}_\mathrm{III}|_{t=t_\mathrm{o}}}{\tilde{\rho}_\mathrm{o}^3} + \frac{3}{4}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_\mathrm{III}|_{t=t_\mathrm{o}}}{\tilde{\rho}_\mathrm{o}^5}\right\}\epsilon^{-1} + O(\epsilon^0). \quad (45)$$

After the derivatives in Eq. (45) are taken, the dependence upon Θ, Φ and r may be removed in favor of $\tilde{\rho}_\mathrm{o}$, χ and δ by use of Eqs. (41)–(43). Then the three steps of (*i*) a Legendre polynomial expansion for the Θ dependence, while r and Φ are held fixed, followed by (*ii*) an integration over Φ, while r is held fixed, and finally (*iii*) taking the limit $\delta \to 0$, together provide the regularization parameters. The techniques involved in the Legendre polynomial expansions and the integration over Φ are described in detail in Appendices C and D of Paper I [7].

Below in Subsections V A and V B, we present the key steps of calculating the A_a and B_a regularization parameters in Eqs. (12)-(19).

A. A_a-terms

We take the ϵ^{-2} term from Eq. (45) and define

$$Q_a[\epsilon^{-2}] \equiv -\frac{q^2}{2}\frac{\partial_a\left(\tilde{\rho}^2\right)|_{t=t_\mathrm{o}}}{\tilde{\rho}_\mathrm{o}^3} \quad (46)$$

Then, we proceed with our calculations of the regularization parameters one component at a time.

1. A_t-term:

First we complete the expression for $Q_t[\epsilon^{-2}]$ by recalling Eqs. (40) and (41)

$$\begin{aligned} Q_t[\epsilon^{-2}] &= -\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}\,\partial_t\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}} \\ &= \frac{q^2}{2}\left[2\left(r_{\rm o}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right)\right]^{-3/2}\left(\frac{2E\dot{r}r_{\rm o}^2\Delta}{f\left(r_{\rm o}^2+J^2\right)}+2EJ\sin\Theta\cos\Phi\right) \\ &= \frac{q^2E\dot{r}r_{\rm o}^2\Delta\chi^{-3/2}}{2\sqrt{2}f\left(r_{\rm o}^2+J^2\right)^{5/2}}\left(\delta^2+1-\cos\Theta\right)^{-3/2} \\ &\quad -\frac{q^2EJ\chi^{-3/2}\cos\Phi}{\sqrt{2}\left(r_{\rm o}^2+J^2\right)^{3/2}}\frac{\partial}{\partial\Theta}\bigg|_{\Delta}\left(\delta^2+1-\cos\Theta\right)^{-1/2}, \end{aligned} \tag{47}$$

where $\partial/\partial\Theta|_{\Delta}$ means that Δ is held constant while the differentiation is performed with respect to Θ.

According to Appendix D of Paper I [7], for $p \geq 1$

$$\left(\delta^2+1-\cos\Theta\right)^{-p-1/2} = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{\delta^{2p-1}(2p-1)}\left[1+O(\ell\delta)\right]P_\ell(\cos\Theta),\ \delta\to 0, \tag{48}$$

and for $p = 0$

$$\left(\delta^2+1-\cos\Theta\right)^{-1/2} = \sum_{\ell=0}^{\infty}\left[\sqrt{2}+O(\ell\delta)\right]P_\ell(\cos\Theta),\ \delta\to 0. \tag{49}$$

Then, by Eqs. (48) for $p = 1$, (49) and (43), in the limit $\delta \to 0$ (equivalently $\Delta \to 0$) Eq. (47) becomes

$$\begin{aligned} \lim_{\Delta\to 0} Q_t[\epsilon^{-2}] &= \mathrm{sgn}(\Delta)\frac{q^2\dot{r}r_{\rm o}\chi^{-1}}{\left(r_{\rm o}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\cos\Theta) \\ &\quad -\frac{q^2EJ\chi^{-3/2}\cos\Phi}{\left(r_{\rm o}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial\Theta}\bigg|_{\Delta}P_\ell(\cos\Theta). \end{aligned} \tag{50}$$

Then, we integrate $\lim_{\Delta\to 0} Q_t[\epsilon^{-2}]$ over Φ and divide it by 2π (we denote this process by the angle brackets "$\langle\,\rangle$")

$$\left\langle\lim_{\Delta\to 0} Q_t[\epsilon^{-2}]\right\rangle = \mathrm{sgn}(\Delta)\frac{q^2\dot{r}r_{\rm o}\left\langle\chi^{-1}\right\rangle}{\left(r_{\rm o}^2+J^2\right)^{3/2}}\sum_{\ell=0}^{\infty}\left(\ell+\frac{1}{2}\right)P_\ell(\cos\Theta), \tag{51}$$

where we exploit the fact that $\left\langle\chi^{-3/2}\cos\Phi\right\rangle = 0$ to get rid of the second term in Eq. (50) [19]. Appendix C of Paper I [7] provides $\langle\chi^{-1}\rangle = {}_2F_1\left(1,\frac{1}{2};1;\alpha\right) \equiv F_1 = (1-\alpha)^{-1/2}$, where $\alpha \equiv J^2/\left(r_{\rm o}^2+J^2\right)$. Substituting this into Eq. (51), the regularization parameter A_t is the coefficient of the sum on the right hand side in the coincidence limit $\Theta \to 0$

$$A_t = \mathrm{sgn}(\Delta)\frac{q^2}{r_{\rm o}^2}\frac{\dot{r}}{1+J^2/r_{\rm o}^2}. \tag{52}$$

2. A_r-term:

Similarly, we have

$$Q_r[\epsilon^{-2}] = -\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}\,\partial_r\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}}. \tag{53}$$

Here, before computing $\partial_r\left(\tilde{\rho}^2\right)|_{t=t_{\rm o}}$ we reverse the steps of Eqs. (34), (36), (39) and (40) to obtain the relation

$$\tilde{\rho}^2 = \mathcal{P}_{\rm II} + O[(x-x_{\rm o})^4], \tag{54}$$

where $\mathcal{P}_{\rm II}$ is now back to Eq. (34). Differentiating this with respect to r and going through the steps of Eqs. (36) and (37), Eq. (53) can be expressed with the help of Eq. (41) as

$$Q_r[\epsilon^{-2}] = -\frac{q^2}{f^2}\left[2\left(r_{\rm o}^2+J^2\right)\chi\left(\delta^2+1-\cos\Theta\right)\right]^{-3/2}\left[\frac{r_{\rm o}^2E^2\Delta}{r_{\rm o}^2+J^2}+fJ\dot{r}\sin\Theta\cos\Phi\right]. \tag{55}$$

Then, the rest of the calculation is carried out in the same fashion as for the case of A_t-term above. We obtain

$$A_r = -\mathrm{sgn}(\Delta)\frac{q^2}{r_{\rm o}^2}\frac{E}{f\left(1+J^2/r_{\rm o}^2\right)}. \tag{56}$$

3. A_ϕ-term:

First we have

$$Q_\phi[\epsilon^{-2}] \;=\; -\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}\,\partial_\phi\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}}. \tag{57}$$

Taking the same steps as used for A_r-term above via Eqs. (54), (36) and (37) in order, we obtain

$$\partial_\phi\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}} = 2\left(r_{\rm o}^2+J^2\right)\sin\Theta\cos\Phi + O[(x-x_{\rm o})^3]. \tag{58}$$

Then, in a similar manner to that employed in the previous cases, in the limit $\Delta\to 0$ Eq. (57) becomes

$$\lim_{\Delta\to 0} Q_\phi[\epsilon^{-2}] = -\frac{q^2\chi^{-3/2}\cos\Phi}{\left(r_{\rm o}^2+J^2\right)^{1/2}}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial\Theta}\bigg|_\Delta P_\ell(\cos\Theta). \tag{59}$$

The right hand side vanishes through "$\langle\,\rangle$" process because $\left\langle\chi^{-3/2}\cos\Phi\right\rangle = 0$. Hence,

$$A_\phi = 0. \tag{60}$$

4. A_θ-term:

It is evident from the particle's motion, which is confined to the equatorial plane $\theta_{\rm o}=\frac{\pi}{2}$, that no self-force is acting on the particle in the direction perpendicular to this plane. This is due to the fact that both the derivatives of retarded field and the singular source field

with respect to θ tend to zero in the coincidence limit. Our calculation of A_θ should support this. Through the same process as employed before, we have

$$Q_\theta[\epsilon^{-2}] = -\frac{q^2}{2}\tilde{\rho}_{\rm o}^{-3}\,\partial_\theta\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}} \tag{61}$$

with

$$\partial_\theta\left(\tilde{\rho}^2\right)\Big|_{t=t_{\rm o}} = 2r_{\rm o}^2\sin\Theta\sin\Phi + O[(x-x_{\rm o})^3]. \tag{62}$$

Then, similarly as in the case of A_ϕ-term above

$$\lim_{\Delta\to 0} Q_\theta[\epsilon^{-2}] = -\frac{q^2 r_{\rm o}^2\chi^{-3/2}\sin\Phi}{(r_{\rm o}^2+J^2)^{3/2}}\sum_{\ell=0}^{\infty}\frac{\partial}{\partial\Theta}\bigg|_{\Delta} P_\ell(\cos\Theta). \tag{63}$$

Again, via "$\langle\,\rangle$" process, the right hand side vanishes because $\left\langle\chi^{-3/2}\sin\Phi\right\rangle = 0$. Thus,

$$A_\theta = 0. \tag{64}$$

B. B_a-terms

We take the ϵ^{-1} term from Eq. (45) and define

$$Q_a[\epsilon^{-1}] \equiv q^2\left\{-\frac{1}{2}\frac{\partial_a\mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left[\partial_a\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}, \tag{65}$$

where for computing $\partial_a\left(\tilde{\rho}^2\right)$, Eq. (54) should be referred to, and $\mathcal{P}_{\rm III}$ is the cubic part taken directly from Eq. (31).

We may express this in a generic form

$$Q_a[\epsilon^{-1}] = \sum_{n=1}^{2}\sum_{k=0}^{2n}\sum_{p=0}^{[k/2]}\frac{b_{nkp(a)}\Delta^{2n-k}\left(\phi-\phi_{\rm o}\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}}, \tag{66}$$

where $\Delta \equiv r - r_{\rm o}$, and $b_{nkp(a)}$ is the coefficient of each individual term that depends on n, k and p as well as a, with a dimension $\mathcal{R}^{k-1}$ for $a = t$, r and $\mathcal{R}^k$ for $a = \theta$, ϕ. We recall from Eqs. (34) and (35) that the first of the steps to lead to $\tilde{\rho}_{\rm o}^2$ in the denominator is replacing $\phi-\phi_{\rm o}$ by $(\phi-\phi')-J\dot{r}\Delta/f(r_{\rm o}^2+J^2)$ to eliminate the coupling term $\Delta\left(\phi-\phi_{\rm o}\right)$. This makes a sum of independent square forms of each of Δ and $\phi-\phi'$, which is a necessary step to induce the Legendre polynomial expansions later. Thus, to be consistent with this modification in the denominator, $\phi-\phi_{\rm o}$ in the numerator on the right hand side of Eq. (66) should be also replaced by $(\phi-\phi')-J\dot{r}\Delta/f(r_{\rm o}^2+J^2)$. Then, this will create a number of additional terms apart from $(\phi-\phi')^m$ when we expand the quantity $[(\phi-\phi')-J\dot{r}\Delta/f(r_{\rm o}^2+J^2)]$ raised, say, to the m-th power, and the computation will be very complicated.

By analyzing the structure of the quantity on the right hand side of Eq. (66) one can prove that $\phi-\phi_{\rm o}$ may be replaced just by $\phi-\phi'$ in the numerator without the term $-J\dot{r}\Delta/f(r_{\rm o}^2+J^2)$ (the same idea is found in Mino, Nakano, and Sasaki [9]). The verification follows. The behavior of the quantity on the right hand side of Eq. (66), according to the powers of each factor, is

$$Q_a[\epsilon^{-1}] \sim \tilde{\rho}_{\rm o}^{-(2n+1)}\Delta^{2n-k}\left(\phi-\phi_{\rm o}\right)^{k-2p}\left(\theta-\frac{\pi}{2}\right)^{2p}\mathcal{R}^s, \tag{67}$$

where $s = k - 1$ for $a = t,\, r$ and $s = k$ for $a = \theta,\, \phi$. Further,

$$
\begin{aligned}
(\phi - \phi_{\rm o})^{k-2p} &= \left[(\phi - \phi') - \frac{J\dot{r}\Delta}{f(r_{\rm o}^2 + J^2)}\right]^{k-2p} \\
&= \sum_{i=0}^{k-2p} c_{kpi}\, (\phi - \phi')^i\, \Delta^{k-2p-i} \sim (\phi - \phi')^i\, \Delta^{k-2p-i}/\mathcal{R}^{k-2p-i} \qquad (68) \\
&\sim (\sin\Theta)^i\, (\cos\Phi)^i\, \Delta^{k-2p-i}/\mathcal{R}^{k-2p-i} + O[(x - x_{\rm o})^{k-2p+2}], \qquad (69)
\end{aligned}
$$

where a binomial expansion over the index $i = 0,\, 1,\, \cdots,\, k - 2p$ is assumed with $c_{kpi} \sim 1/\mathcal{R}^{k-2p-i}$ in Eq. (68), and in Eq. (69) $(\phi - \phi')^i$ is replaced by $[\sin(\phi - \phi')]^i + O[(\phi - \phi')^{i+2}]$ — the term $O[(x - x_{\rm o})^{k-2p+2}]$ at the end results from this $O[(\phi - \phi')^{i+2}]$, then the coordinates are rotated using the definition of new angles by Eq. (37). Also, by Eq. (37) again

$$
\left(\theta - \frac{\pi}{2}\right)^{2p} = (\sin\Theta)^{2p}\, (\sin\Phi)^{2p} + O[(x - x_{\rm o})^{2p+2}]. \qquad (70)
$$

Using Eqs. (69) and (70), the behavior of $Q[\epsilon^{-1}]$ in Eq. (67) looks like

$$
Q_a[\epsilon^{-1}] \sim \tilde{\rho}_{\rm o}^{-(2n+1)} \Delta^{2n-2p-i}\, (\sin\Theta)^{2p+i}\, (\cos\Phi)^i\, (\sin\Phi)^{2p}\, \mathcal{R}^s, \qquad (71)
$$

where $s = 2p + i - 1$ for $a = t,\, r$ and $s = 2p + i$ for $a = \theta,\, \phi$, and any contributions from $O[(x - x_{\rm o})^{k-2p+2}]$ in Eq. (69) and from $O[(x - x_{\rm o})^{2p+2}]$ in Eq. (70) have been disregarded: by putting these pieces into Eq. (67) we simply obtain ϵ^1-terms, which would correspond to $O(\ell^{-2})$ in Eq. (11) and should vanish when summed over ℓ in our final self-force calculation by Eq. (10). $Q_a[\epsilon^{-1}]$ then can be categorized into the following cases:

(i) $i = 2j + 1$ $(j = 0,\, 1,\, 2,\, \cdots)$
The integrand for "$\langle\,\rangle$" process, $F(\Phi) \equiv (\cos\Phi)^{2j+1}\, (\sin\Phi)^{2p}$ has the property $F(\Phi + \pi) = -F(\Phi)$. Thus

$$
\left\langle Q_a[\epsilon^{-1}] \right\rangle = 0, \qquad (72)
$$

(ii) $i = 2j$ $(j = 0,\, 1,\, 2,\, \cdots)$
Using Eqs. (41) and (43), we can express $(\sin\Theta)^{2p+i}$ in Eq. (71) above in terms of $\tilde{\rho}_{\rm o}$ and Δ via a binomial expansion

$$
\begin{aligned}
(\sin\Theta)^{2p+2j} &= [2\,(1 - \cos\Theta)]^{p+j} + O[(x - x_{\rm o})^{2(p+j)+2}] \\
&= \sum_{q=0}^{p+j} d_{pjq} \tilde{\rho}_{\rm o}^{2q} \Delta^{2(p+j-q)} + O[(x - x_{\rm o})^{2(p+j)+2}] \qquad (73) \\
&\sim \tilde{\rho}_{\rm o}^{2q} \Delta^{2(p+j-q)}/\mathcal{R}^{2(p+j)} + O[(x - x_{\rm o})^{2(p+j)+2}], \qquad (74)
\end{aligned}
$$

where $q = 0,\, 1,\, \cdots,\, p + j$ is the index for a binomial expansion and $d_{pjq} \sim 1/\mathcal{R}^{2(p+j)}$. When Eq. (74) is substituted into Eq. (71), the contribution from $O[(x - x_{\rm o})^{2(p+j)+2}]$ can be disregarded since it would correspond to $O(\epsilon^1)$ again. Then, we have

$$
Q_a[\epsilon^{-1}] \sim (\sin\Phi)^{2p}\, (\cos\Phi)^{2j}\, \tilde{\rho}_{\rm o}^{-2(n-q)-1} \Delta^{2(n-q)} \mathcal{R}^s, \qquad (75)
$$

where $s = -1$ for $a = t,\, r$ and $s = 0$ for $a = \theta,\, \phi$, and we can guarantee that $n - q \geq 0$ always since $0 \leq q \leq p + j = p + \frac{1}{2}i$, $0 \leq i \leq k - 2p$ and $p \leq k \leq 2n$. Then, Eq. (75) can be subcategorized into the following two cases;

(ii-1) $n - q \geq 1$
By Eqs. (41), (43) and (48)

$$Q_a[\epsilon^{-1}] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p} (\cos\Phi)^{2j} \, \Delta P_\ell(\cos\Theta)\mathcal{R}^s \longrightarrow 0, \tag{76}$$

(ii-2) $n - q = 0$
By Eqs. (41), (43) and (49)

$$Q_a[\epsilon^{-1}] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p} (\cos\Phi)^{2j} \, P_\ell(\cos\Theta)\mathcal{R}^s, \tag{77}$$

where $s = -1$ for $a = t$, r and $s = 0$ for $a = \theta$, ϕ.

Therefore, by analyzing the structure of $Q_a[\epsilon^{-1}]$ we find that the ϵ^{-1}-terms vanish in all the cases except when $n - q = 0$. The non-vanishing B_a-terms are derived only from this case. Then, by $0 \leq q \leq p + j = p + \frac{1}{2}i$, $0 \leq i \leq k - 2p$ and $p \leq k \leq 2n$ together with $n = q$ one can show that

$$0 \ \leq \ k - 2p - i \text{ and } k - 2p - i \leq 0, \text{ i.e. } k - 2p - i = 0. \tag{78}$$

Substituting this result into Eq. (68), then into Eq. (66) we may conclude that in the numerator of $Q[\epsilon^{-1}]$ in Eq. (66) one can simply substitute

$$(\phi - \phi_{\rm o})^{k-2p} \rightarrow (\phi - \phi')^{k-2p} . \text{ Q. E. D.} \tag{79}$$

The significance of this proof does not lie in the result given by Eq. (79) only, but also in the fact that the non-vanishing contribution comes only from the case $n = q$ for Eq. (75), i.e.

$$Q_a[\epsilon^{-1}] \sim (\sin\Phi)^{2p} (\cos\Phi)^{2(n-p)} \, \tilde{\rho}_{\rm o}^{-1}\mathcal{R}^s, \tag{80}$$

where $n = 1$, 2 and $0 \leq p \leq n$, and $s = -1$ for $a = t$, r and $s = 0$ for $a = \theta$, ϕ.

Below are presented the calculations of B_a-terms of the regularization parameters by component, in a similar manner to those for A_a-terms.

1. B_t-term:

We begin with

$$Q_t[\epsilon^{-1}] = q^2 \left\{ -\frac{1}{2} \frac{\partial_t \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4} \frac{\left[\partial_t\left(\tilde{\rho}^2\right)\right] \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \right\}. \tag{81}$$

The subsequent computation will be very lengthy and it will be reasonable to split $Q_t[\epsilon^{-1}]$ into two parts. First, let

$$Q_{t(1)}[\epsilon^{-1}] \equiv -\frac{q^2}{2} \tilde{\rho}_{\rm o}^{-3} \, \partial_t \mathcal{P}_{\rm III}|_{t=t_{\rm o}} \, , \tag{82}$$

where

$$\partial_t \mathcal{P}_{\mathrm{III}}\big|_{t=t_{\mathrm{o}}} = -\frac{ME\dot{r}\Delta^2}{f^2 r_{\mathrm{o}}^2} - 2\left(1 - \frac{M}{r_{\mathrm{o}}}\right)\frac{EJ\Delta}{fr_{\mathrm{o}}}\left(\phi - \phi_{\mathrm{o}}\right) + r_{\mathrm{o}} E\dot{r}\left[\left(\phi - \phi_{\mathrm{o}}\right)^2 + \left(\theta - \frac{\pi}{2}\right)^2\right]. \quad (83)$$

As proved at the beginning of this Subsection, every $(\phi - \phi_{\mathrm{o}})^m$ in the numerators of the ϵ^{-1}-term can be replaced by $(\phi - \phi')^m$ without affecting the rest of calculation. Then, followed by the rotation of the coordinates via Eq. (37)

$$\begin{aligned} Q_{t(1)}[\epsilon^{-1}] \;=\; & -\frac{q^2}{2}\tilde{\rho}_{\mathrm{o}}^{-3}\left[-\frac{ME\dot{r}\Delta^2}{f^2 r_{\mathrm{o}}^2} - 2\left(1 - \frac{M}{r_{\mathrm{o}}}\right)\frac{EJ\Delta}{fr_{\mathrm{o}}}\sin\Theta\cos\Phi + 2r_{\mathrm{o}}E\dot{r}\left(1 - \cos\Theta\right)\right] \\ & + O\left[\frac{(x - x_{\mathrm{o}})^4}{\tilde{\rho}_{\mathrm{o}}^3}\right], \end{aligned} \quad (84)$$

where an approximation $\sin^2\Theta = 2(1 - \cos\Theta) + O[(x - x_{\mathrm{o}})^4]$ is used to obtain the last term inside the first bracket. Here we may drop off the term $O\left[(x - x_{\mathrm{o}})^4/\tilde{\rho}_{\mathrm{o}}^3\right]$, which is essentially $O(\epsilon^1)$, for the same reason as explained at the beginning of this subsection. Then, using the same techniques as used to find A_a-terms, we can reduce Eq. (84) to

$$\begin{aligned} Q_{t(1)}[\epsilon^{-1}] \;=\; & \left[\frac{q^2 ME\dot{r}}{2f^2 r_{\mathrm{o}}^2} + \frac{q^2 r_{\mathrm{o}}^3 E^3 \dot{r}\chi^{-1}}{2f^2\left(r_{\mathrm{o}}^2 + J^2\right)^2}\right]\Delta^2\left[2\left(r_{\mathrm{o}}^2 + J^2\right)\chi\left(\delta^2 + 1 - \cos\Theta\right)\right]^{-3/2} \\ & - \frac{q^2\left(1 - \frac{M}{r_{\mathrm{o}}}\right)EJ\Delta\chi^{-3/2}\cos\Phi}{\sqrt{2}fr_{\mathrm{o}}\left(r_{\mathrm{o}}^2 + J^2\right)^{3/2}}\left.\frac{\partial}{\partial\Theta}\right|_{\Delta}\left(\delta^2 + 1 - \cos\Theta\right)^{-1/2} \\ & - \frac{q^2 E\dot{r} r_{\mathrm{o}}\chi^{-1}}{2\left(r_{\mathrm{o}}^2 + J^2\right)}\tilde{\rho}_{\mathrm{o}}^{-1}. \end{aligned} \quad (85)$$

As we have seen before, by Eq. (48) $\left(\delta^2 + 1 - \cos\Theta\right)^{-3/2} \sim \Delta^{-1}$ in the limit $\Delta \to 0$ and the first term on the right hand side will vanish. The second term will also give no contribution to the regularization parameters because $\left\langle\chi^{-3/2}\cos\Phi\right\rangle = 0$. Only the last term, which is $\sim \tilde{\rho}_{\mathrm{o}}^{-1}$, will give non-zero contribution according to the argument in the analysis presented above (see Eq. (80)). Using Eq. (49) in the limit $\Delta \to 0$ and taking "$\langle\,\rangle$" process, Eq. (85) becomes

$$\left\langle\lim_{\Delta\to 0} Q_{t(1)}[\epsilon^{-1}]\right\rangle = -\frac{1}{2}\frac{q^2}{r_{\mathrm{o}}^2}\frac{E\dot{r}\left\langle\chi^{-3/2}\right\rangle}{\left(1 + J^2/r_{\mathrm{o}}^2\right)^{3/2}}\sum_{\ell=0}^{\infty} P_\ell\left(\cos\Theta\right). \quad (86)$$

The identity $\langle\chi^{-p}\rangle \equiv \left\langle\left(1 - \alpha\sin^2\Phi\right)^{-p}\right\rangle = {}_2F_1\left(p,\, \frac{1}{2};\, 1,\, \alpha\right) \equiv F_p$, with $\alpha \equiv J^2/\left(r_{\mathrm{o}}^2 + J^2\right)$ is taken from Appendix C of Paper I [7], and we take the limit $\Theta \to 0$

$$\left.\left\langle\lim_{\Delta\to 0} Q_{t(1)}[\epsilon^{-1}]\right\rangle\right|_{\Theta\to 0} = -\frac{1}{2}\frac{q^2}{r_{\mathrm{o}}^2}\frac{E\dot{r}F_{3/2}}{\left(1 + J^2/r_{\mathrm{o}}^2\right)^{3/2}}. \quad (87)$$

Now the remaining part is

$$Q_{t(2)}[\epsilon^{-1}] \equiv \frac{3q^2}{4}\tilde{\rho}_{\mathrm{o}}^{-5}\left[\partial_t\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\mathrm{III}}\Big|_{t=t_{\mathrm{o}}}, \quad (88)$$

where

$$\left[\partial_t\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\Big|_{t=t_{\rm o}} = \left[-\frac{2E\dot{r}\Delta}{f} - 2EJ\left(\phi-\phi_{\rm o}\right)\right]$$
$$\times\left[-\left(1+\frac{\dot{r}^2}{f}\right)\frac{M\Delta^3}{f^2 r_{\rm o}^2} + \left(2-\frac{5M}{r_{\rm o}}\right)\frac{J\dot{r}\Delta^2}{f^2 r_{\rm o}}\left(\phi-\phi_{\rm o}\right)\right.$$
$$+\left(1-\frac{\dot{r}}{f}+\frac{2J^2}{r_{\rm o}^2}\right)r_{\rm o}\Delta\left(\phi-\phi_{\rm o}\right)^2 + \left(1-\frac{\dot{r}^2}{f}\right)r_{\rm o}\Delta\left(\theta-\frac{\pi}{2}\right)^2$$
$$\left.-r_{\rm o}J\dot{r}\left(\phi-\phi_{\rm o}\right)^3 - r_{\rm o}J\dot{r}\left(\phi-\phi_{\rm o}\right)\left(\theta-\frac{\pi}{2}\right)^2\right] + O[(x-x_{\rm o})^6]. \quad (89)$$

Taking similar procedures as above, the non-vanishing contributions turn out to be

$$\left\langle \lim_{\Delta\to 0} Q_{t(2)}[\epsilon^{-1}]\right\rangle = \left\langle \lim_{\Delta\to 0}\frac{3}{2}q^2 EJ^2\dot{r}r_{\rm o}\tilde{\rho}_{\rm o}^{-5}\cos^2\Phi\sin^4\Theta\right\rangle$$
$$= \left\langle \lim_{\Delta\to 0}\frac{3}{2}\frac{q^2}{r_{\rm o}}\frac{E\dot{r}\tilde{\rho}_{\rm o}^{-1}}{1+J^2/r_{\rm o}^2}\left(\chi^{-1}-\frac{\chi^{-2}}{1+J^2/r_{\rm o}^2}\right)\right\rangle$$
$$= \frac{3}{2}\frac{q^2}{r_{\rm o}^2}\frac{E\dot{r}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\left(\left\langle\chi^{-3/2}\right\rangle - \frac{\left\langle\chi^{-5/2}\right\rangle}{1+J^2/r_{\rm o}^2}\right)\sum_{\ell=0}^{\infty}P_\ell\left(\cos\Theta\right), \quad (90)$$

where all other terms than $\sim\tilde{\rho}_{\rm o}^{-1}$ again have been dropped off during the procedure since they vanish either in the limit $\Delta\to 0$ or through the "$\langle\,\rangle$" process. Then, using the identity $\langle\chi^{-p}\rangle \equiv {}_2F_1\left(p,\,\frac{1}{2};\,1,\,\alpha\right) \equiv F_p$, we have

$$\left\langle \lim_{\Delta\to 0} Q_{t(2)}[\epsilon^{-1}]\right\rangle\Big|_{\Theta\to 0} = \frac{3}{2}\frac{q^2}{r_{\rm o}^2}\frac{E\dot{r}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}}\left(F_{3/2}-\frac{F_{5/2}}{1+J^2/r_{\rm o}^2}\right). \quad (91)$$

By combining Eqs. (87) and (91), we finally obtain

$$B_t = \frac{q^2}{r_{\rm o}^2}E\dot{r}\left[\frac{F_{3/2}}{\left(1+J^2/r_{\rm o}^2\right)^{3/2}} - \frac{3F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right]. \quad (92)$$

2. B_r-term:

From Eq. (65) we start with

$$Q_r[\epsilon^{-1}] = q^2\left\{-\frac{1}{2}\frac{\partial_r\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4}\frac{\left[\partial_r\left(\tilde{\rho}^2\right)\right]\mathcal{P}_{\rm III}\big|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5}\right\}. \quad (93)$$

Then, following the same steps as taken for the case of B_t-term above, we obtain

$$B_r = \frac{q^2}{r_{\rm o}^2}\left[-\frac{F_{1/2}}{\left(1+J^2/r_{\rm o}^2\right)^{1/2}} + \frac{\left(1-2f^{-1}\dot{r}^2\right)F_{3/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{3/2}} + \frac{3f^{-1}\dot{r}^2F_{5/2}}{2\left(1+J^2/r_{\rm o}^2\right)^{5/2}}\right]. \quad (94)$$

3. B_ϕ-term:

Again, from Eq. (65)

$$Q_\phi[\epsilon^{-1}] = q^2 \left\{ -\frac{1}{2} \frac{\partial_\phi \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4} \frac{[\partial_\phi (\tilde{\rho}^2)]\, \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \right\}. \tag{95}$$

Then, similarly we can derive

$$B_\phi = \frac{q^2}{J} \dot{r} \left[\frac{F_{1/2} - F_{3/2}}{\left(1 + J^2/r_{\rm o}^2\right)^{1/2}} + \frac{3(F_{5/2} - F_{3/2})}{2\left(1 + J^2/r_{\rm o}^2\right)^{3/2}} \right]. \tag{96}$$

4. B_θ-term:

As A_θ vanishes, so should B_θ. From

$$Q_\theta[\epsilon^{-1}] = q^2 \left\{ -\frac{1}{2} \frac{\partial_\theta \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4} \frac{[\partial_\theta (\tilde{\rho}^2)]\, \mathcal{P}_{\rm III}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \right\}, \tag{97}$$

one finds that there is no term like $\sim \tilde{\rho}_{\rm o}^{-1}$: all terms are either like $\sim \Delta^{2n}/\tilde{\rho}_{\rm o}^{2n+1}$ or like $\sim \Delta^{2n-1} \sin\Theta \cos\Phi / \tilde{\rho}_{\rm o}^{2n+1}$ ($n = 1, 2$), which vanish in the limit $\Delta \to 0$ or through the "$\langle\,\rangle$" process. Thus

$$B_\theta = 0. \tag{98}$$

C. C_a-terms

We have mentioned before that C_a-terms, which originate from ϵ^0-term in Eq. (45), always vanish. This can be proved by analyzing the structure of ϵ^0-term. First we specify the ϵ^0-order term for $\partial_a(1/\rho)|_{t=t_{\rm o}}$ in a Laurent series expansion and define

$$\begin{aligned} Q_a[\epsilon^0] \;\equiv\; q^2 \Bigg\{ & -\frac{1}{2} \frac{\partial_a \mathcal{P}_{\rm IV}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^3} + \frac{3}{4} \frac{(\partial_a \mathcal{P}_{\rm III})\, \mathcal{P}_{\rm III}|_{t=t_{\rm o}} + [\partial_a (\tilde{\rho}^2)]\, \mathcal{P}_{\rm IV}|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^5} \\ & -\frac{15}{16} \frac{[\partial_a (\tilde{\rho}^2)]\, \mathcal{P}_{\rm III}^2|_{t=t_{\rm o}}}{\tilde{\rho}_{\rm o}^7} \Bigg\}. \end{aligned} \tag{99}$$

Generically, this can be written as

$$Q_a[\epsilon^0] = \sum_{n=1}^{3} \sum_{k=0}^{2n+1} \sum_{p=0}^{[k/2]} \frac{c_{nkp(a)} \Delta^{2n+1-k} \left(\phi - \phi_{\rm o}\right)^{k-2p} \left(\theta - \frac{\pi}{2}\right)^{2p}}{\tilde{\rho}_{\rm o}^{2n+1}}, \tag{100}$$

where $\Delta \equiv r - r_{\rm o}$, and $c_{nkp(a)}$ is the coefficient of each individual term that depends on n, k and p as well as a, with a dimension $\mathcal{R}^{k-2}$ for $a = t$, r and $\mathcal{R}^{k-1}$ for $a = \theta$, ϕ.

The behavior of $Q_a[\epsilon^0]$, according to the powers of each factor on the right hand side of Eq. (100), is

$$Q_a[\epsilon^0] \sim \tilde{\rho}_{\rm o}^{-(2n+1)} \Delta^{2n+1-k} \left(\phi - \phi_{\rm o}\right)^{k-2p} \left(\theta - \frac{\pi}{2}\right)^{2p} \mathcal{R}^s, \tag{101}$$

where $s = k - 2$ for $a = t$, r and $s = k - 1$ for $a = \theta$, ϕ. Following the same procedure as in the beginning of Subsection V B, Eq. (101) becomes

$$Q_a[\epsilon^0] \sim \tilde{\rho}_{\rm o}^{-(2n+1)} \Delta^{2n+1-2p-i} (\sin\Theta)^{2p+i} (\sin\Phi)^{2p} (\cos\Phi)^i \mathcal{R}^s, \tag{102}$$

where a binomial expansion over the index $i = 0, 1, \cdots, k-2p$ is assumed, and $s = 2p+i-2$ for $a = t$, r and $s = 2p + i - 1$ for $a = \theta$, ϕ. Here we have disregarded any by-products like $O[(x - x_{\rm o})^{k-2p+2}]$ and $O[(x - x_{\rm o})^{2p+2}]$, which originate from $(\phi - \phi_{\rm o})^{k-2p}$ and $\left(\theta - \frac{\pi}{2}\right)^{2p}$, respectively when we rotate the angles: by putting them back into Eq. (101) we simply obtain ϵ^2-terms, which would correspond to $O(\ell^{-4})$ in Eq. (11) and should vanish when summed over ℓ in our final self-force calculation by Eq. (10). Then, the rest of the argument is developed in the same way as in the beginning of Subsection V B:

(i) $i = 2j + 1$ $(j = 0, 1, 2, \cdots)$
The integrand for "$\langle\,\rangle$" process, $F(\Phi) \equiv (\cos\Phi)^{2j+1} (\sin\Phi)^{2p}$ has the property $F(\Phi + \pi) = -F(\Phi)$. Thus

$$\left\langle Q_a[\epsilon^0] \right\rangle = 0, \tag{103}$$

(ii) $i = 2j$ $(j = 0, 1, 2, \cdots)$
We have

$$Q_a[\epsilon^0] \sim (\sin\Phi)^{2p} (\cos\Phi)^{2j} \tilde{\rho}_{\rm o}^{-2(n-q)-1} \Delta^{2(n-q)+1} \mathcal{R}^s, \tag{104}$$

where $q = 0, 1, \cdots, p + j$ is the index for a binomial expansion and $s = -2$ for $a = t$, r and $s = -1$ for $a = \theta$, ϕ. Here we can guarantee that $n - q \geq -\frac{1}{2}$, i.e. $n - q = 0, 1, 2, \cdots$ since $0 \leq q \leq p + j = p + \frac{1}{2}i$, $0 \leq i \leq k - 2p$ and $p \leq k \leq 2n + 1$. Then, Eq. (104) can be subcategorized into the following two cases;

(ii-1) $n - q \geq 1$
By Eqs. (41), (43) and (48)

$$Q_a[\epsilon^0] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p} (\cos\Phi)^{2j} \Delta^2 P_\ell(\cos\Theta) \mathcal{R}^s \longrightarrow 0, \tag{105}$$

(ii-2) $n - q = 0$
By Eqs. (41), (43) and (49)

$$Q_a[\epsilon^0] \underset{\Delta\to 0}{\sim} (\sin\Phi)^{2p} (\cos\Phi)^{2j} \Delta P_\ell(\cos\Theta) \mathcal{R}^s \longrightarrow 0, \tag{106}$$

where $s = -2$ for $a = t$, r and $s = -1$ for $a = \theta$, ϕ.

Clearly, in any cases the quantity $Q_a[\epsilon^0]$ does not survive, therefore we can conclude that C_a-terms are always zero. Q. E. D.

Also, this justifies the argument that we need not clarify the term $O[(x - x_{\rm o})^3]$ in Eq. (23) and its contribution to ρ^2, which is $O[(x - x_{\rm o})^4]$ in Eqs. (30) and (31) in Section IV or $\mathcal{P}_{\rm IV}$ in Eqs. (33) and (44) in Section V: by the analysis of the generic structure given above, $-\frac{1}{2}\, \partial_a \mathcal{P}_{\rm IV}|_{t=t_{\rm o}} / \tilde{\rho}_{\rm o}^3$ or $\frac{3}{4}\, [\partial_a (\tilde{\rho}^2)]\, \mathcal{P}_{\rm IV}|_{t=t_{\rm o}} / \tilde{\rho}_{\rm o}^5$ would simply vanish in the coincidence limit $x \longrightarrow x_{\rm o}$, regardless of what $\mathcal{P}_{\rm IV}$ is.

VI. DISCUSSION

Self-force analysis in curved spacetime relies upon the ability to divide the field of a point charge into two parts. One part is singular and exerts no net force on the charge itself. The remainder is a regular, smooth vacuum field and is entirely the cause of any self-force. We see, in this manuscript, that the singular field is adequately approximated by its Coulomb field in a coordinate system which is locally inertial and centered upon the charge. With this elementary approximation of the singular field, it is guaranteed that the remainder is at least differentiable and provides the correct self-force. For a charge moving in the Schwarzschild geometry, the multipole moments of the singular field are the regularization parameters which are necessary for computing the self-force from a multipole expansion, and these regularization parameters are calculated with a relatively elementary procedure. Our analysis agrees with that of others [9, 10] and appears to us to be the most straightforward calculation of these important parameters.

Future work will use a higher order approximation for the singular field which will result in a more accurate and more differentiable approximation for the regular remainder. In practice, a higher order approximation of the self-force significantly speeds up convergence in a mode sum, as demonstrated in Paper I [7].

The simplicity of our methods should also make them useful for self-force analysis in the context of the Kerr geometry.

Another avenue for future work involves calculating gravitational self-force regularization parameters. Thus far, published parameters [9, 10] focus upon the self-force, rather than upon the the metric perturbations themselves. However, it is clear that a gravitational self-force calculation of a gauge invariant quantity requires the metric perturbations explicitly as well.

Future work will find the THZ coordinates to higher orders, and calculate higher order regularization parameters which will provide faster convergence of the ℓ sums and correspondingly more accurate results.

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APPENDIX A: HYPERGEOMETRIC FUNCTIONS AND REPRESENTATIONS OF REGULARIZATION PARAMETERS

In Section V we define

$$\chi \equiv 1 - \alpha \sin^2 \Phi \tag{A1}$$

with

$$\alpha \equiv \frac{J^2}{r_o^2 + J^2}. \tag{A2}$$

And we use

$$\left\langle \chi^{-p} \right\rangle \equiv \left\langle \left(1-\alpha\sin^2\Phi\right)^{-p} \right\rangle = \frac{2}{\pi}\int_0^{\pi/2} \left(1-\alpha\sin^2\Phi\right)^{-p} d\Phi$$
$$= {}_2F_1\left(p, \frac{1}{2}; 1, \alpha\right) \equiv F_p. \tag{A3}$$

In particular, for the cases $p = \frac{1}{2}$ and $p = -\frac{1}{2}$ we have the following representations

$$F_{1/2} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1; \alpha\right) = \frac{2}{\pi}\hat{K}(\alpha) \tag{A4}$$

and

$$F_{-1/2} = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1; \alpha\right) = \frac{2}{\pi}\hat{E}(\alpha), \tag{A5}$$

where $\hat{K}(\alpha)$ and $\hat{E}(\alpha)$ are called complete elliptic integrals of the first and second kinds, respectively.

If we take the derivative of $F_{1/2}$ with respect to $k \equiv \sqrt{\alpha}$ via Eq. (A3), we obtain

$$\frac{\partial F_{1/2}}{\partial k} = -\frac{F_{1/2}}{k} + \frac{F_{3/2}}{k}, \tag{A6}$$

or using Eq. (A4)

$$\frac{\partial \hat{K}}{\partial k} = -\frac{\hat{K}}{k} + \frac{\pi}{2}\frac{F_{3/2}}{k}. \tag{A7}$$

However, Ref. [18] shows that

$$\frac{\partial \hat{K}}{\partial k} = \frac{\hat{E}}{k\left(1-k^2\right)} - \frac{\hat{K}}{k}. \tag{A8}$$

Thus, by comparing Eq. (A7) and Eq. (A8) we find the representation

$$F_{3/2} = \frac{2}{\pi}\frac{\hat{E}}{1-k^2} = \frac{2}{\pi}\frac{\hat{E}}{1-\alpha}. \tag{A9}$$

Further, we can also find the representation for $F_{5/2}$. First, taking the derivative of $F_{3/2}$ with respect to $k \equiv \sqrt{\alpha}$ via Eq. (A3) gives

$$\frac{\partial F_{3/2}}{\partial k} = -\frac{3F_{3/2}}{k} + \frac{3F_{5/2}}{k}. \tag{A10}$$

Also, using Eq. (A9) together with Eqs. (A3)-(A5), another expression for the same derivative is obtained solely in terms of complete elliptic integrals

$$\frac{\partial F_{3/2}}{\partial k} = \frac{2}{\pi}\frac{\left(1+k^2\right)\hat{E} - \left(1-k^2\right)\hat{K}}{k\left(1-k^2\right)^2}. \tag{A11}$$

Then, by Eqs. (A9), (A10), and, (A11) we find

$$F_{5/2} = \frac{2}{3\pi}\left[\frac{2\left(2-\alpha\right)\hat{E}}{\left(1-\alpha\right)^2} - \frac{\hat{K}}{1-\alpha}\right]. \tag{A12}$$

Now, using Eqs. (A4), (A9) and (A12), we may rewrite the non-zero B_a regularization parameters, Eqs. (15)-(17) in Section III as

$$B_t = \frac{q^2}{r_\mathrm{o}^2} \frac{E\dot{r}\left[\hat{K}(\alpha) - 2\hat{E}(\alpha)\right]}{\pi\,(1 + J^2/r_\mathrm{o}^2)^{3/2}}, \tag{A13}$$

$$B_r = \frac{q^2}{r_\mathrm{o}^2} \frac{(\dot{r}^2 - 2E^2)\,\hat{K}(\alpha) + (\dot{r}^2 + E^2)\,\hat{E}(\alpha)}{\pi\,(1 - 2M/r_\mathrm{o})\,(1 + J^2/r_\mathrm{o}^2)^{3/2}}, \tag{A14}$$

$$B_\phi = \frac{q^2}{r_\mathrm{o}} \frac{\dot{r}\left[\hat{K}(\alpha) - \hat{E}(\alpha)\right]}{\pi\,(J/r_\mathrm{o})\,(1 + J^2/r_\mathrm{o}^2)^{1/2}}, \tag{A15}$$

which are exactly the same to the results of Barack and Ori [10].

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[19] Or alternatively, one can use the argument $\frac{\partial}{\partial\Theta}\Big|_\Delta P_\ell(\cos\Theta) = 0$ as $\Theta \to 0$, to show that this part does not survive at the end.